UNITED STATES DEPARTMENT OF COMMERCE • Luther H. Hodges, *Secretary*
NATIONAL BUREAU OF STANDARDS • A. V. Astin, *Director*

Handbook of Mathematical Functions

With

Formulas, Graphs, and Mathematical Tables

Edited by
Milton Abramowitz and Irene A. Stegun



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The text relating to physical constants and conversion factors (page 6) has been modified to take into account the newly adopted Système International d'Unites (SI).

ERRATA NOTICE

The original printing of this Handbook (June 1964) contained errors that have been corrected in the reprinted editions. These corrections are marked with an asterisk (*) for identification. The errors occurred on the following pages: 2-3, 6-8, 10, 15, 19-20, 25, 76, 85, 91, 102, 187, 189-197, 218, 223, 225, 233, 250, 255, 260-263, 268, 271-273, 292, 302, 328, 332, 333-337, 362, 365, 415, 423, 438-440, 443, 445, 447, 449, 451, 484, 498, 505-506, 509-510, 543, 556, 558, 562, 571, 595, 599, 600, 722-723, 739, 742, 744, 746, 752, 756, 760-765, 774, 777-785, 790, 797, 801, 822-823, 832, 835, 844, 886-889, 897, 914, 915, 920, 930-931, 936, 940-941, 944-950, 953, 960, 963, 989-990, 1010, 1026.

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Preface

The present volume is an outgrowth of a Conference on Mathematical Tables held at Cambridge, Mass., on September 15-16, 1954, under the auspices of the National Science Foundation and the Massachusetts Institute of Technology. The purpose of the meeting was to evaluate the need for mathematical tables in the light of the availability of large scale computing machines. It was the consensus of opinion that in spite of the increasing use of the new machines the basic need for tables would continue to exist.

Numerical tables of mathematical functions are in continual demand by scientists and engineers. A greater variety of functions and higher accuracy of tabulation are now required as a result of scientific advances and, especially, of the increasing use of automatic computers. In the latter connection, the tables serve mainly for preliminary surveys of problems before programming for machine operation. For those without easy access to machines, such tables are, of course, indispensable.

Consequently, the Conference recognized that there was a pressing need for a modernized version of the classical tables of functions of Jahnke-Emde. To implement the project, the National Science Foundation requested the National Bureau of Standards to prepare such a volume and established an Ad Hoc Advisory Committee, with Professor Philip M. Morse of the Massachusetts Institute of Technology as chairman, to advise the staff of the National Bureau of Standards during the course of its preparation. In addition to the Chairman, the Committee consisted of A. Erdelyi, M. C. Gray, N. Metropolis, J. B. Rosser, H. C. Thacher, Jr., John Todd, C. B. Tompkins, and J. W. Tukey.

The primary aim has been to include a maximum of useful information within the limits of a moderately large volume, with particular attention to the needs of scientists in all fields. An attempt has been made to cover the entire field of special functions. To carry out the goal set forth by the Ad Hoc Committee, it has been necessary to supplement the tables by including the mathematical properties that are important in computation work, as well as by providing numerical methods which demonstrate the use and extension of the tables.

The Handbook was prepared under the direction of the late Milton Abramowitz, and Irene A. Stegun. Its success has depended greatly upon the cooperation of many mathematicians. Their efforts together with the cooperation of the Ad Hoc Committee are greatly appreciated. The particular contributions of these and other individuals are acknowledged at appropriate places in the text. The sponsorship of the National Science Foundation for the preparation of the material is gratefully recognized.

It is hoped that this volume will not only meet the needs of all table users but will in many cases acquaint its users with new functions.

ALLEN V. ASTIN, *Director*

June 1964
Washington, D.C.

Preface to the Ninth Printing

The enthusiastic reception accorded the "Handbook of Mathematical Functions" is little short of unprecedented in the long history of mathematical tables that began when John Napier published his tables of logarithms in 1614. Only four and one-half years after the first copy came from the press in 1964, Myron Tribus, the Assistant Secretary of Commerce for Science and Technology, presented the 100,000th copy of the Handbook to Lee A. DuBridge, then Science Advisor to the President. Today, total distribution is approaching the 150,000 mark at a scarcely diminished rate.

The success of the Handbook has not ended our interest in the subject. On the contrary, we continue our close watch over the growing and changing world of computation and to discuss with outside experts and among ourselves the various proposals for possible extension or supplementation of the formulas, methods and tables that make up the Handbook.

In keeping with previous policy, a number of errors discovered since the last printing have been corrected. Aside from this, the mathematical tables and accompanying text are unaltered. However, some noteworthy changes have been made in Chapter 2: Physical Constants and Conversion Factors, pp. 6-8. The table on page 7 has been revised to give the values of physical constants obtained in a recent reevaluation; and pages 6 and 8 have been modified to reflect changes in definition and nomenclature of physical units and in the values adopted for the acceleration due to gravity in the revised Potsdam system.

The record of continuing acceptance of the Handbook, the praise that has come from all quarters, and the fact that it is one of the most-quoted scientific publications in recent years are evidence that the hope expressed by Dr. Astin in his Preface is being amply fulfilled.

LEWIS M. BRANSCOMB, *Director*
National Bureau of Standards

November 1970

Foreword

This volume is the result of the cooperative effort of many persons and a number of organizations. The National Bureau of Standards has long been turning out mathematical tables and has had under consideration, for at least 10 years, the production of a compendium like the present one. During a Conference on Tables, called by the NBS Applied Mathematics Division on May 15, 1952, Dr. Abramowitz of that Division mentioned preliminary plans for such an undertaking, but indicated the need for technical advice and financial support.

The Mathematics Division of the National Research Council has also had an active interest in tables; since 1943 it has published the quarterly journal, "Mathematical Tables and Aids to Computation" (MTAC), editorial supervision being exercised by a Committee of the Division.

Subsequent to the NBS Conference on Tables in 1952 the attention of the National Science Foundation was drawn to the desirability of financing activity in table production. With its support a 2-day Conference on Tables was called at the Massachusetts Institute of Technology on September 15–16, 1954, to discuss the needs for tables of various kinds. Twenty-eight persons attended, representing scientists and engineers using tables as well as table producers. This conference reached consensus on several conclusions and recommendations, which were set forth in the published Report of the Conference. There was general agreement, for example, "that the advent of high-speed computing equipment changed the task of table making but definitely did not remove the need for tables". It was also agreed that "an outstanding need is for a Handbook of Tables for the Occasional Computer, with tables of usually encountered functions and a set of formulas and tables for interpolation and other techniques useful to the occasional computer". The Report suggested that the NBS undertake the production of such a Handbook and that the NSF contribute financial assistance. The Conference elected, from its participants, the following Committee: P. M. Morse (Chairman), M. Abramowitz, J. H. Curtiss, R. W. Hamming, D. H. Lehmer, C. B. Tompkins, J. W. Tukey, to help implement these and other recommendations.

The Bureau of Standards undertook to produce the recommended tables and the National Science Foundation made funds available. To provide technical guidance to the Mathematics Division of the Bureau, which carried out the work, and to provide the NSF with independent judgments on grants for the work, the Conference Committee was reconstituted as the Committee on Revision of Mathematical Tables of the Mathematics Division of the National Research Council. This, after some changes of membership, became the Committee which is signing this Foreword. The present volume is evidence that Conferences can sometimes reach conclusions and that their recommendations sometimes get acted on.

Active work was started at the Bureau in 1956. The overall plan, the selection of authors for the various chapters, and the enthusiasm required to begin the task were contributions of Dr. Abramowitz. Since his untimely death, the effort has continued under the general direction of Irene A. Stegun. The workers at the Bureau and the members of the Committee have had many discussions about content, style and layout. Though many details have had to be argued out as they came up, the basic specifications of the volume have remained the same as were outlined by the Massachusetts Institute of Technology Conference of 1954.

The Committee wishes here to register its commendation of the magnitude and quality of the task carried out by the staff of the NBS Computing Section and their expert collaborators in planning, collecting and editing these Tables, and its appreciation of the willingness with which its various suggestions were incorporated into the plans. We hope this resulting volume will be judged by its users to be a worthy memorial to the vision and industry of its chief architect, Milton Abramowitz. We regret he did not live to see its publication.

P. M. Morse, *Chairman.*
A. Erdélyi
M. C. Gray
N. C. Metropolis
J. B. Rosser
H. C. Thacher, Jr.
John Todd
C. B. Tompkins
J. W. Tukey.

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Edited by Milton Abramowitz and Irene A. Stegun

1. Introduction

The present Handbook has been designed to provide scientific investigators with a comprehensive and self-contained summary of the mathematical functions that arise in physical and engineering problems. The well-known Tables of Functions by E. Jahnke and F. Emde has been invaluable to workers in these fields in its many editions[1] during the past half-century. The present volume extends the work of these authors by giving more extensive and more accurate numerical tables, and by giving larger collections of mathematical properties of the tabulated functions. The number of functions covered has also been increased.

The classification of functions and organization of the chapters in this Handbook is similar to that of An Index of Mathematical Tables by A. Fletcher, J. C. P. Miller, and L. Rosenhead.[2] In general, the chapters contain numerical tables, graphs, polynomial or rational approximations for automatic computers, and statements of the principal mathematical properties of the tabulated functions, particularly those of computational importance. Many numerical examples are given to illustrate the use of the tables and also the computation of function values which lie outside their range. At the end of the text in each chapter there is a short bibliography giving books and papers in which proofs of the mathematical properties stated in the chapter may be found. Also listed in the bibliographies are the more important numerical tables. Comprehensive lists of tables are given in the Index mentioned above, and current information on new tables is to be found in the National Research Council quarterly Mathematics of Computation (formerly Mathematical Tables and Other Aids to Computation).

The mathematical notations used in this Handbook are those commonly adopted in standard texts, particularly Higher Transcendental Functions, Volumes 1–3, by A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi (McGraw-Hill, 1953–55). Some alternative notations have also been listed. The introduction of new symbols has been kept to a minimum, and an effort has been made to avoid the use of conflicting notation.

2. Accuracy of the Tables

The number of significant figures given in each table has depended to some extent on the number available in existing tabulations. There has been no attempt to make it uniform throughout the Handbook, which would have been a costly and laborious undertaking. In most tables at least five significant figures have been provided, and the tabular intervals have generally been chosen to ensure that linear interpolation will yield four- or five-figure accuracy, which suffices in most physical applications. Users requiring higher precision in their interpolates may obtain them by use of higher-order interpolation procedures, described below.

In certain tables many-figured function values are given at irregular intervals in the argument. An example is provided by Table 9.4. The purpose of these tables is to furnish "key values" for the checking of programs for automatic computers; no question of interpolation arises.

The maximum end-figure error, or "tolerance" in the tables in this Handbook is $\frac{1}{2}$ of 1 unit everywhere in the case of the elementary functions, and 1 unit in the case of the higher functions except in a few cases where it has been permitted to rise to 2 units.

[1] The most recent, the sixth, with F. Loesch added as co-author, was published in 1960 by McGraw-Hill, U.S.A., and Teubner, Germany.

[2] The second edition, with L. J. Comrie added as co-author, was published in two volumes in 1962 by Addison-Wesley, U.S.A., and Scientific Computing Service Ltd., Great Britain.

3. Auxiliary Functions and Arguments

One of the objects of this Handbook is to provide tables or computing methods which enable the user to evaluate the tabulated functions over complete ranges of real values of their parameters. In order to achieve this object, frequent use has been made of auxiliary functions to remove the infinite part of the original functions at their singularities, and auxiliary arguments to cope with infinite ranges. An example will make the procedure clear.

The exponential integral of positive argument is given by

$$\mathrm{Ei}(x)=\int_{-\infty}^{x}\frac{e^u}{u}du$$

$$=\gamma+\ln x+\frac{x}{1\cdot 1!}+\frac{x^2}{2\cdot 2!}+\frac{x^3}{3\cdot 3!}+\ \cdots$$

$$\sim\frac{e^x}{x}\left[1+\frac{1!}{x}+\frac{2!}{x^2}+\frac{3!}{x^3}+\ \cdots\right](x\to\infty)$$

The logarithmic singularity precludes direct interpolation near $x=0$. The functions $\mathrm{Ei}(x)-\ln x$ and $x^{-1}[\mathrm{Ei}(x)-\ln x-\gamma]$, however, are well-behaved and readily interpolable in this region. Either will do as an auxiliary function; the latter was in fact selected as it yields slightly higher accuracy when $\mathrm{Ei}(x)$ is recovered. The function $x^{-1}[\mathrm{Ei}(x)-\ln x-\gamma]$ has been tabulated to nine decimals for the range $0\leq x\leq\frac{1}{2}$. For $\frac{1}{2}\leq x\leq 2$, $\mathrm{Ei}(x)$ is sufficiently well-behaved to admit direct tabulation, but for larger values of x, its exponential character predominates. A smoother and more readily interpolable function for large x is $xe^{-x}\mathrm{Ei}(x)$; this has been tabulated for $2\leq x\leq 10$. Finally, the range $10\leq x\leq\infty$ is covered by use of the inverse argument x^{-1}. Twenty-one entries of $xe^{-x}\mathrm{Ei}(x)$, corresponding to $x^{-1}=.1(-.005)0$, suffice to produce an interpolable table.

4. Interpolation

The tables in this Handbook are not provided with differences or other aids to interpolation, because it was felt that the space they require could be better employed by the tabulation of additional functions. Admittedly aids could have been given without consuming extra space by increasing the intervals of tabulation, but this would have conflicted with the requirement that linear interpolation is accurate to four or five figures.

For applications in which linear interpolation is insufficiently accurate it is intended that Lagrange's formula or Aitken's method of iterative linear interpolation[3] be used. To help the user, there is a statement at the foot of most tables of the maximum error in a linear interpolate, and the number of function values needed in Lagrange's formula or Aitken's method to interpolate to full tabular accuracy.

As an example, consider the following extract from Table 5.1.

x	$xe^xE_1(x)$	x	$xe^xE_1(x)$
7.5	.89268 7854	8.0	.89823 7113
7.6	.89384 6312	8.1	.89927 7888
7.7	.89497 9666	8.2	.90029 7306
7.8	.89608 8737	8.3	.90129 60°3
7.9	.89717 4302	8.4	.90227 4695

$$\left[\begin{matrix}(-6)3\\5\end{matrix}\right]$$

The numbers in the square brackets mean that the maximum error in a linear interpolate is 3×10^{-6}, and that to interpolate to the full tabular accuracy five points must be used in Lagrange's and Aitken's methods.

[3] A. C. Aitken, On interpolation by iteration of proportional parts, without the use of differences, Proc. Edinburgh Math. Soc. 3, 56–76 (1932).

Let us suppose that we wish to compute the value of $xe^xE_1(x)$ for $x=7.9527$ from this table. We describe in turn the application of the methods of linear interpolation, Lagrange and Aitken, and of alternative methods based on differences and Taylor's series.

(1) Linear interpolation. The formula for this process is given by

$$f_p=(1-p)f_0+pf_1$$

where f_0, f_1 are consecutive tabular values of the function, corresponding to arguments x_0, x_1, respectively; p is the given fraction of the argument interval

$$p=(x-x_0)/(x_1-x_0)$$

and f_p the required interpolate. In the present instance, we have

$$f_0=.89717\ 4302 \qquad f_1=.89823\ 7113 \qquad p=.527$$

The most convenient way to evaluate the formula on a desk calculating machine is to set f_0 and f_1 in turn on the keyboard, and carry out the multiplications by $1-p$ and p cumulatively; a partial check is then provided by the multiplier dial reading unity. We obtain

$$f_{.527}=(1-.527)(.89717\ 4302)+.527(.89823\ 7113)$$
$$=.89773\ 4403.$$

Since it is known that there is a possible error of 3×10^{-6} in the linear formula, we round off this result to .89773. The maximum possible error in this answer is composed of the error committed

by the last rounding, that is, $.4403\times10^{-5}$, plus 3×10^{-6}, and so certainly cannot exceed $.8\times10^{-5}$.

(2) Lagrange's formula. In this example, the relevant formula is the 5-point one, given by

$$f=A_{-2}(p)f_{-2}+A_{-1}(p)f_{-1}+A_0(p)f_0+A_1(p)f_1+A_2(p)f_2$$

Tables of the coefficients $A_k(p)$ are given in chapter 25 for the range $p=0(.01)1$. We evaluate the formula for $p=.52$, $.53$ and $.54$ in turn. Again, in each evaluation we accumulate the $A_k(p)$ in the multiplier register since their sum is unity. We now have the following subtable.

x	$xe^xE_1(x)$		
7.952	.89772 9757		
		10622	
7.953	.89774 0379		−2
		10620	
7.954	.89775 0999		

The numbers in the third and fourth columns are the first and second differences of the values of $xe^xE_1(x)$ (see below); the smallness of the second difference provides a check on the three interpolations. The required value is now obtained by linear interpolation:

$$f_p=.3(.89772\ 9757)+.7(.89774\ 0379)$$
$$=.89773\ 7192.$$

In cases where the correct order of the Lagrange polynomial is not known, one of the preliminary interpolations may have to be performed with polynomials of two or more different orders as a check on their adequacy.

(3) Aitken's method of iterative linear interpolation. The scheme for carrying out this process in the present example is as follows:

n	x_n	$y_n=xe^xE_1(x)$	$y_{0,n}$	$y_{0,1,n}$	$y_{0,1,2,n}$	$y_{0,1,2,3,n}$	x_n-x
0	8.0	.89823 7113					.0473
1	7.9	.89717 4302	.89773 44034				−.0527
2	8.1	.89927 7888	.89774 48264	.89773 71499			.1473
3	7.8	.89608 8737	2 90220	2394	.89773 71938		−.1527
4	8.2	.90029 7306	4 98773	1216	16	89773 71930	.2473
5	7.7	.89497 9666	2 35221	2706	43	30	−.2527

Here

$$y_{0,n}=\frac{1}{x_n-x_0}\begin{vmatrix} y_0 & x_0-x \\ y_n & x_n-x \end{vmatrix}$$

$$y_{0,1,n}=\frac{1}{x_n-x_1}\begin{vmatrix} y_{0,1} & x_1-x \\ y_{0,n} & x_n-x \end{vmatrix}$$

$$y_{0,1,\ldots,m-1,m,n}=\frac{1}{x_n-x_m}\begin{vmatrix} y_{0,1,\ldots,m-1,m} & x_m-x \\ y_{0,1,\ldots,m-1,n} & x_n-x \end{vmatrix}$$

If the quantities x_n-x and x_m-x are used as multipliers when forming the cross-product on a desk machine, their accumulation $(x_n-x)-(x_m-x)$ in the multiplier register is the divisor to be used at that stage. An extra decimal place is usually carried in the intermediate interpolates to safeguard against accumulation of rounding errors.

The order in which the tabular values are used is immaterial to some extent, but to achieve the maximum rate of convergence and at the same time minimize accumulation of rounding errors, we begin, as in this example, with the tabular argument nearest to the given argument, then take the nearest of the remaining tabular arguments, and so on.

The number of tabular values required to achieve a given precision emerges naturally in the course of the iterations. Thus in the present example six values were used, even though it was known in advance that five would suffice. The extra row confirms the convergence and provides a valuable check.

(4) Difference formulas. We use the central difference notation (chapter 25),

x_0	f_0				
		$\delta f_{1/2}$			
x_1	f_1		$\delta^2 f_1$		
		$\delta f_{3/2}$		$\delta^3 f_{3/2}$	
x_2	f_2		$\delta^2 f_2$		$\delta^4 f_2$
		$\delta f_{5/2}$		$\delta^3 f_{5/2}$	
x_3	f_3		$\delta^2 f_3$		
		$\delta f_{7/2}$			
x_4	f_4				

Here

$$\delta f_{1/2}=f_1-f_0,\ \delta f_{3/2}=f_2-f_1,\ \ldots,$$
$$\delta^2 f_1=\delta f_{3/2}-\delta f_{1/2}=f_2-2f_1+f_0$$
$$\delta^3 f_{3/2}=\delta^2 f_2-\delta^2 f_1=f_3-3f_2+3f_1-f_0$$
$$\delta^4 f_2=\delta^3 f_{5/2}-\delta^3 f_{3/2}=f_4-4f_3+6f_2-4f_1+f_0$$

and so on.

In the present example the relevant part of the difference table is as follows, the differences being written in units of the last decimal place of the function, as is customary. The smallness of the high differences provides a check on the function values

x	$xe^xE_1(x)$	$\delta^2 f$	$\delta^4 f$
7.9	.89717 4302	−2 2754	−34
8.0	.89823 7113	−2 2036	−39

Applying, for example, Everett's interpolation formula

$$f_p=(1-p)f_0+E_2(p)\delta^2 f_0+E_4(p)\delta^4 f_0+\ \ldots$$
$$+pf_1+F_2(p)\delta^2 f_1+F_4(p)\delta^4 f_1+\ \ldots$$

and taking the numerical values of the interpolation coefficients $E_2(p)$, $E_4(p)$, $F_2(p)$ and $F_4(p)$ from Table 25.1, we find that

$$10^9 f_{.527} = .473(89717\ 4302) + .061196(2\ 2754) - .012(34)$$
$$+ .527(89823\ 7113) + .063439(2\ 2036) - .012(39)$$
$$= 89773\ 7193.$$

We may notice in passing that Everett's formula shows that the error in a linear interpolate is approximately

$$E_2(p)\delta^2 f_0 + F_2(p)\delta^2 f_1 \approx \tfrac{1}{2}[E_2(p) + F_2(p)][\delta^2 f_0 + \delta^2 f_1]$$

Since the maximum value of $|E_2(p)+F_2(p)|$ in the range $0<p<1$ is $\frac{1}{8}$, the maximum error in a linear interpolate is approximately

$$\frac{1}{16}|\delta^2 f_0 + \delta^2 f_1|, \text{ that is, } \frac{1}{16}|f_2 - f_1 - f_0 + f_{-1}|.$$

(5) Taylor's series. In cases where the successive derivatives of the tabulated function can be computed fairly easily, Taylor's expansion

$$f(x) = f(x_0) + (x-x_0)\frac{f'(x_0)}{1!} + (x-x_0)^2\frac{f''(x_0)}{2!} + (x-x_0)^3\frac{f'''(x_0)}{3!} + \cdots$$

can be used. We first compute as many of the derivatives $f^{(n)}(x_0)$ as are significant, and then evaluate the series for the given value of x. An advisable check on the computed values of the derivatives is to reproduce the adjacent tabular values by evaluating the series for $x=x_{-1}$ and x_1.

In the present example, we have

$$f(x) = xe^x E_1(x)$$
$$f'(x) = (1+x^{-1})f(x) - 1$$
$$f''(x) = (1+x^{-1})f'(x) - x^{-2}f(x)$$
$$f'''(x) = (1+x^{-1})f''(x) - 2x^{-2}f'(x) + 2x^{-3}f(x).$$

With $x_0=7.9$ and $x-x_0=.0527$ our computations are as follows; an extra decimal has been retained in the values of the terms in the series to safeguard against accumulation of rounding errors.

k	$f^{(k)}(x_0)/k!$	$(x-x_0)^k f^{(k)}(x_0)/k!$
0	.89717 4302	.89717 4302
1	.01074 0669	.00056 6033 3
2	−.00113 7621	−.00000 3159 5
3	.00012 1987	.00000 0017 9
		.89773 7194

5. Inverse Interpolation

With linear interpolation there is no difference in principle between direct and inverse interpolation. In cases where the linear formula provides an insufficiently accurate answer, two methods are available. We may interpolate directly, for example, by Lagrange's formula to prepare a new table at a fine interval in the neighborhood of the approximate value, and then apply accurate inverse linear interpolation to the subtabulated values. Alternatively, we may use Aitken's method or even possibly the Taylor's series method, with the roles of function and argument interchanged.

It is important to realize that the accuracy of an inverse interpolate may be very different from that of a direct interpolate. This is particularly true in regions where the function is slowly varying, for example, near a maximum or minimum. The maximum precision attainable in an inverse interpolate can be estimated with the aid of the formula

$$\Delta x \approx \Delta f / \frac{df}{dx}$$

in which Δf is the maximum possible error in the function values.

Example. Given $xe^x E_1(x) = .9$, find x from the table on page X.

(i) Inverse linear interpolation. The formula for p is

$$p = (f_p - f_0)/(f_1 - f_0).$$

In the present example, we have

$$p = \frac{.9 - .89927\ 7888}{.90029\ 7306 - .89927\ 7888} = \frac{72\ 2112}{101\ 9418} = .708357.$$

The desired x is therefore

$$x = x_0 + p(x_1 - x_0) = 8.1 + .708357(.1) = 8.17083\ 57$$

To estimate the possible error in this answer, we recall that the maximum error of direct linear interpolation in this table is $\Delta f = 3\times10^{-6}$. An approximate value for df/dx is the ratio of the first difference to the argument interval (chapter 25), in this case .010. Hence the maximum error in x is approximately $3\times10^{-6}/(.010)$, that is, .0003.

(ii) Subtabulation method. To improve the approximate value of x just obtained, we interpolate directly for $p=.70$, .71 and .72 with the aid of Lagrange's 5-point formula,

x	$xe^x E_1(x)$	δ	δ^2
8.170	.89999 3683		
		1 0151	
8.171	.90000 3834		−2
		1 0149	
8.172	.90001 3983		

Inverse linear interpolation in the new table gives

$$p = \frac{.9 - .89999\ 3683}{.00001\ 0151} = .6223$$

Hence $x = 8.17062\ 23$.

An estimate of the maximum error in this result is

$$\Delta f / \frac{df}{dx} \approx \frac{1\times10^{-9}}{.010} = 1\times10^{-7}$$

(iii) Aitken's method. This is carried out in the same manner as in direct interpolation.

n	$y_n=xe^xE_1(x)$	x_n	$x_{0,n}$	$x_{0,1,n}$	$x_{0,1,2,n}$	$x_{0,1,2,3,n}$	y_n-y
0	.90029 7306	8.2					.00029 7306
1	.89927 7888	8.1	8.17083 5712				−.00072 2112
2	.90129 6033	8.3	8.17023 1505	8.17061 9521			.00129 6033
3	.89823 7113	8.0	8.17113 8043	2 5948	8.17062 2244		−.00176 2887
4	.90227 4695	8.4	8.16992 9437	1 7335	415	8.17062 2318	.00227 4695
5	.89717 4302	7.9	8.17144 0382	2 8142	231	265	−.00282 5698

The estimate of the maximum error in this result is the same as in the subtabulation method. An indication of the error is also provided by the discrepancy in the highest interpolates, in this case $x_{0,1,2,3,4}$, and $x_{0,1,2,3,5}$.

6. Bivariate Interpolation

Bivariate interpolation is generally most simply performed as a sequence of univariate interpolations. We carry out the interpolation in one direction, by one of the methods already described, for several tabular values of the second argument in the neighborhood of its given value. The interpolates are differenced as a check, and interpolation is then carried out in the second direction.

An alternative procedure in the case of functions of a complex variable is to use the Taylor's series expansion, provided that successive derivatives of the function can be computed without much difficulty.

7. Generation of Functions from Recurrence Relations

Many of the special mathematical functions which depend on a parameter, called their index, order or degree, satisfy a linear difference equation (or recurrence relation) with respect to this parameter. Examples are furnished by the Legendre function $P_n(x)$, the Bessel function $J_n(x)$ and the exponential integral $E_n(x)$, for which we have the respective recurrence relations

$$(n+1)P_{n+1}-(2n+1)xP_n+nP_{n-1}=0$$

$$J_{n+1}-\frac{2n}{x}J_n+J_{n-1}=0$$

$$nE_{n+1}+xE_n=e^{-x}.$$

Particularly for automatic work, recurrence relations provide an important and powerful computing tool. If the values of $P_n(x)$ or $J_n(x)$ are known for two consecutive values of n, or $E_n(x)$ is known for one value of n, then the function may be computed for other values of n by successive applications of the relation. Since generation is carried out perforce with rounded values, it is vital to know how errors may be propagated in the recurrence process. If the errors do not grow relative to the size of the wanted function, the process is said to be stable. If, however, the relative errors grow and will eventually overwhelm the wanted function, the process is unstable.

It is important to realize that stability may depend on (i) the particular solution of the difference equation being computed; (ii) the values of x or other parameters in the difference equation; (iii) the direction in which the recurrence is being applied. Examples are as follows.

Stability—increasing n

$P_n(x)$, $P_n^m(x)$

$Q_n(x)$, $Q_n^m(x)$ $(x<1)$

$Y_n(x)$, $K_n(x)$

$J_{-n-\frac{1}{2}}(x)$, $I_{-n-\frac{1}{2}}(x)$

$E_n(x)$ $(n<x)$

Stability—decreasing n

$P_n(x)$, $P_n^m(x)$ $(x<1)$

$Q_n(x)$, $Q_n^m(x)$

$J_n(x)$, $I_n(x)$

$J_{n+\frac{1}{2}}(x)$, $I_{n+\frac{1}{2}}(x)$

$E_n(x)$ $(n>x)$

$F_n(\eta, \rho)$ (Coulomb wave function)

Illustrations of the generation of functions from their recurrence relations are given in the pertinent chapters. It is also shown that even in cases where the recurrence process is unstable, it may still be used when the starting values are known to sufficient accuracy.

Mention must also be made here of a refinement, due to J. C. P. Miller, which enables a recurrence process which is stable for decreasing n to be applied without any knowledge of starting values for large n. Miller's algorithm, which is well-suited to automatic work, is described in **19.28, Example 1.**

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M. Abramowitz.

12. Struve Functions and Related Functions

MILTON ABRAMOWITZ[1]

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[1] National Bureau of Standards. (Deceased.)

12. Struve Functions and Related Functions

Mathematical Properties

12.1. Struve Function $\mathbf{H}_\nu(z)$

Differential Equation and General Solution

12.1.1

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+(z^2-\nu^2)w=\frac{4(\frac{1}{2}z)^{\nu+1}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}$$

The general solution is

12.1.2 $w=aJ_\nu(z)+bY_\nu(z)+\mathbf{H}_\nu(z)$ $(a,b$, constants$)$

where $z^{-\nu}\mathbf{H}_\nu(z)$ is an entire function of z.

Power Series Expansion

12.1.3

$$\mathbf{H}_\nu(z)=(\tfrac{1}{2}z)^{\nu+1}\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{1}{2}z)^{2k}}{\Gamma(k+\frac{3}{2})\Gamma(k+\nu+\frac{3}{2})}$$

12.1.4 $$\mathbf{H}_0(z)=\frac{2}{\pi}\left[z-\frac{z^3}{1^2\cdot3^2}+\frac{z^5}{1^2\cdot3^2\cdot5^2}-\cdots\right]$$

12.1.5

$$\mathbf{H}_1(z)=\frac{2}{\pi}\left[\frac{z^2}{1^2\cdot3}-\frac{z^4}{1^2\cdot3^2\cdot5}+\frac{z^6}{1^2\cdot3^2\cdot5^2\cdot7}-\cdots\right]$$

Integral Representations

If $\mathscr{R}\nu>-\frac{1}{2}$,

12.1.6

$$\mathbf{H}_\nu(z)=\frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^1(1-t^2)^{\nu-\frac{1}{2}}\sin(zt)\,dt$$

12.1.7 $$=\frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^{\frac{\pi}{2}}\sin(z\cos\theta)\sin^{2\nu}\theta\,d\theta$$

12.1.8 $$=Y_\nu(z)+\frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^\infty e^{-zt}(1+t^2)^{\nu-\frac{1}{2}}dt \qquad \left(|\arg z|<\frac{\pi}{2}\right)$$

Recurrence Relations

12.1.9 $$\mathbf{H}_{\nu-1}+\mathbf{H}_{\nu+1}=\frac{2\nu}{z}\mathbf{H}_\nu+\frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})}$$

12.1.10 $$\mathbf{H}_{\nu-1}-\mathbf{H}_{\nu+1}=2\mathbf{H}'_\nu-\frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})}$$

12.1.11 $$\mathbf{H}'_0=(2/\pi)-\mathbf{H}_1$$

12.1.12 $$\frac{d}{dz}(z^\nu\mathbf{H}_\nu)=z^\nu\mathbf{H}_{\nu-1}$$

12.1.13 $$\frac{d}{dz}(z^{-\nu}\mathbf{H}_\nu)=\frac{1}{\sqrt{\pi}\,2^\nu\Gamma(\nu+\frac{3}{2})}-z^{-\nu}\mathbf{H}_{\nu+1}$$

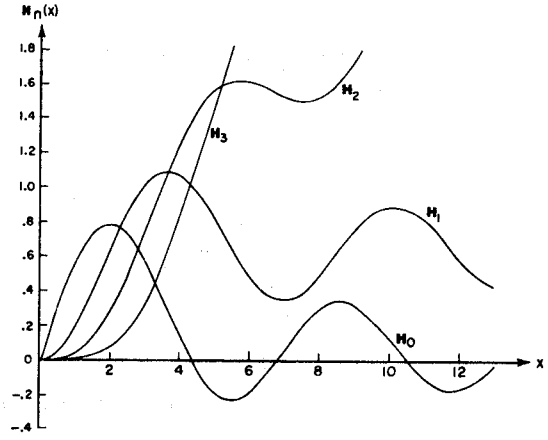


FIGURE 12.1. *Struve functions.*

$\mathbf{H}_n(x)$, $n=0(1)3$

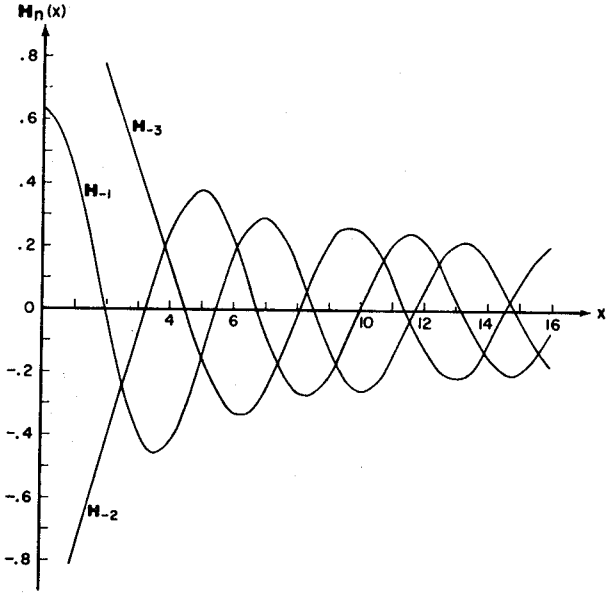


FIGURE 12.2. *Struve functions.*

$\mathbf{H}_n(x)$, $-n=1(1)3$

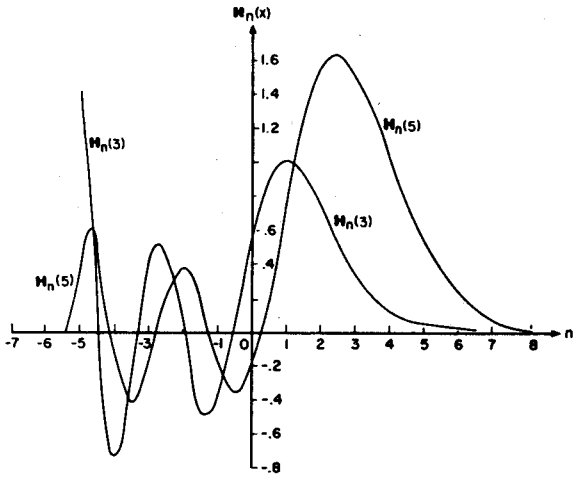


FIGURE 12.3. *Struve functions.*

$\mathbf{H}_n(x)$, $x=3, 5$

Special Properties

12.1.14 $\mathbf{H}_\nu(x)\geq 0 \qquad (x>0 \text{ and } \nu\geq\frac{1}{2})$

12.1.15

$$\mathbf{H}_{-(n+\frac{1}{2})}(z)=(-1)^n J_{n+\frac{1}{2}}(z) \quad (n \text{ an integer}\geq 0)$$

12.1.16 $$\mathbf{H}_{\frac{1}{2}}(z)=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}(1-\cos z)$$

12.1.17

$$\mathbf{H}_{\frac{3}{2}}(z)=\left(\frac{z}{2\pi}\right)^{\frac{1}{2}}\left(1+\frac{2}{z^2}\right)-\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\left(\sin z+\frac{\cos z}{z}\right)$$

12.1.18 $$\mathbf{H}_\nu(ze^{m\pi i})=e^{m(\nu+1)\pi i}\mathbf{H}_\nu(z) \ (m \text{ an integer})$$

12.1.19 $$\mathbf{H}_0(z)=\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{J_{2k+1}(z)}{2k+1}$$

12.1.20 $$\mathbf{H}_1(z)=\frac{2}{\pi}-\frac{2}{\pi}J_0(z)+\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{J_{2k}(z)}{4k^2-1}$$

12.1.21 $$\mathbf{H}_\nu(z)=\frac{2(z/2)^{\nu+1}}{\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})}\,{}_1F_2\left(1;\frac{3}{2}+\nu,\frac{3}{2};-\frac{z^2}{4}\right)$$

Integrals (See chapter 11)

12.1.22 $$\int_0^\infty t^{-1}\mathbf{H}_0(t)dt=\frac{\pi}{2}$$

12.1.23

$$\int_0^z \mathbf{H}_0(t)dt=\frac{2}{\pi}\left[\frac{z^2}{2}-\frac{z^4}{1^2\cdot 3^2\cdot 4}+\frac{z^6}{1^2\cdot 3^2\cdot 5^2\cdot 6}-\cdots\right]$$

12.1.24 $$\int_0^z t^{-\nu}\mathbf{H}_{\nu+1}(t)dt=\frac{z}{2^\nu\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})}-z^{-\nu}\mathbf{H}_\nu(z)$$

Struve's Integral

12.1.25

$$\frac{4}{\pi}\int_z^\infty t^{-2}\mathbf{H}_1(t)dt=\frac{2}{\pi z}\mathbf{H}_1(z)+\frac{2}{\pi}\int_z^\infty t^{-1}\mathbf{H}_0(t)dt$$

12.1.26

$$\frac{2}{\pi}\int_z^\infty t^{-1}\mathbf{H}_0(t)dt=1-\frac{4}{\pi^2}\left[z-\frac{z^3}{1^2\cdot 3^2\cdot 3}+\frac{z^5}{1^2\cdot 3^2\cdot 5^2\cdot 5}-\cdots\right]$$

12.1.27

$$\int_0^\infty t^{\mu-\nu-1}\mathbf{H}_\nu(t)dt=\frac{\Gamma(\frac{1}{2}\mu)2^{\mu-\nu-1}\tan(\frac{1}{2}\pi\mu)}{\Gamma(\nu-\frac{1}{2}\mu+1)} \quad (|\mathscr{R}\mu|<1, \mathscr{R}\nu>\mathscr{R}\mu-\tfrac{3}{2})$$

If $f_\nu(z)=\int_0^z \mathbf{H}_\nu(t)t^\nu dt$

12.1.28

$$f_{\nu+1}=(2\nu+1)f_\nu(z)-z^{\nu+1}\mathbf{H}_\nu(z)+\frac{z^{2\nu+2}}{(\nu+1)2^{\nu+1}\Gamma(\frac{1}{2})\Gamma(\nu+\frac{3}{2})} \quad (\mathscr{R}\nu>-\tfrac{1}{2})$$

Asymptotic Expansions for Large $|z|$

12.1.29

$$\mathbf{H}_\nu(z)-Y_\nu(z)=\frac{1}{\pi}\sum_{k=0}^{m-1}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(\nu+\frac{1}{2}-k)\left(\frac{z}{2}\right)^{2k-\nu+1}}+R_m \quad (|\arg z|<\pi)$$

where $R_m=O(|z|^{\nu-2m-1})$. If ν is real, z positive * and $m+\frac{1}{2}-\nu\geq 0$, the remainder after m terms is of the same sign and numerically less than the first term neglected.

12.1.30

$$\mathbf{H}_0(z)-Y_0(z)\sim\frac{2}{\pi}\left[\frac{1}{z}-\frac{1}{z^3}+\frac{1^2\cdot 3^2}{z^5}-\frac{1^2\cdot 3^2\cdot 5^2}{z^7}+\cdots\right] \quad (|\arg z|<\pi)$$

12.1.31

$$\mathbf{H}_1(z)-Y_1(z)\sim\frac{2}{\pi}\left[1+\frac{1}{z^2}-\frac{1^2\cdot 3}{z^4}+\frac{1^2\cdot 3^2\cdot 5}{z^6}-\cdots\right] \quad (|\arg z|<\pi)$$

12.1.32

$$\int_0^z[\mathbf{H}_0(t)-Y_0(t)]dt-\frac{2}{\pi}[\ln(2z)+\gamma]\sim\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(2k)!(2k-1)!}{(k!)^2(2z)^{2k}} \quad (|\arg z|<\pi)$$

where $\gamma=.57721\ 56649\ \ldots$ is Euler's constant.

12.1.33

$$\int_z^\infty t^{-1}[\mathbf{H}_0(t)-Y_0(t)]dt\sim\frac{2}{\pi z}\sum_{k=0}^{\infty}\frac{(-1)^k[(2k)!]^2}{(k!)^2(2k+1)(2z)^{2k}} \quad (|\arg z|<\pi)$$

*See page 11.

Asymptotic Expansions for Large Orders

12.1.34

$$\mathbf{H}_\nu(z)-Y_\nu(z)\sim\frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\sum_{k=0}^{\infty}\frac{k!\,b_k}{z^{k+1}}$$

$(|\arg z|<\frac{1}{2}\pi, |\nu|<|z|)$

$b_0=1,\ b_1=2\nu/z,\ b_2=6(\nu/z)^2-\frac{1}{2},\ b_3=20(\nu/z)^3-4(\nu/z)$

12.1.35

$$\mathbf{H}_\nu(z)+iJ_\nu(z)\sim\frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\sum_{k=0}^{\infty}\frac{k!\,b_k}{z^{k+1}}\qquad(|\nu|>|z|)$$

12.2. Modified Struve Function $\mathbf{L}_\nu(z)$

Power Series Expansion

12.2.1

$$\mathbf{L}_\nu(z)=-ie^{-\frac{i\nu\pi}{2}}\mathbf{H}_\nu(iz)$$

$$=(\tfrac{1}{2}z)^{\nu+1}\sum_{k=0}^{\infty}\frac{(z/2)^{2k}}{\Gamma(k+\frac{3}{2})\Gamma(k+\nu+\frac{3}{2})}$$

Integral Representations

12.2.2

$$\mathbf{L}_\nu(z)=\frac{2(z/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^{\frac{\pi}{2}}\sinh(z\cos\theta)\sin^{2\nu}\theta\,d\theta$$

$(\mathscr{R}\nu>-\frac{1}{2})$

12.2.3

$$I_{-\nu}(x)-\mathbf{L}_\nu(x)=\frac{2(x/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^{\infty}\sin(tx)(1+t^2)^{\nu-\frac{1}{2}}\,dt$$

$(\mathscr{R}\nu<\frac{1}{2}, x>0)$

Recurrence Relations

12.2.4

$$\mathbf{L}_{\nu-1}-\mathbf{L}_{\nu+1}=\frac{2\nu}{z}\mathbf{L}_\nu+\frac{(z/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})}$$

12.2.5

$$\mathbf{L}_{\nu-1}+\mathbf{L}_{\nu+1}=2\mathbf{L}'_\nu-\frac{(z/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})}$$

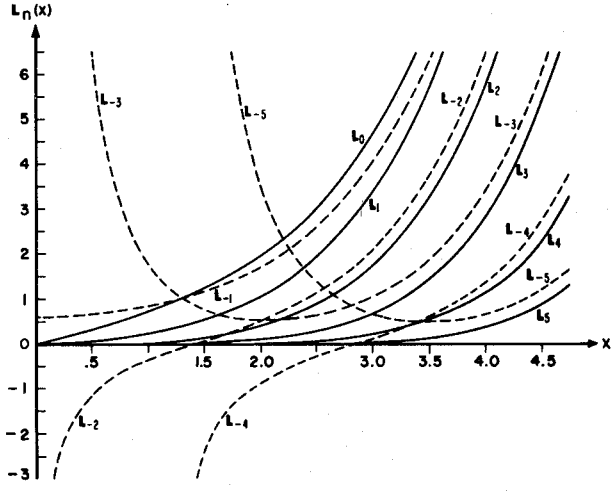


FIGURE 12.4. *Modified Struve functions.*

$\mathbf{L}_n(x),\ \pm n=0(1)5$

Asymptotic Expansion for Large $|z|$

12.2.6

$$\mathbf{L}_\nu(z)-I_{-\nu}(z)$$

$$\sim\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}\Gamma(k+\frac{1}{2})}{\Gamma(\nu+\frac{1}{2}-k)\left(\frac{z}{2}\right)^{2k-\nu+1}}\quad(|\arg z|<\tfrac{1}{2}\pi)$$

Integrals

12.2.7

$$\int_0^z\mathbf{L}_0(t)\,dt=\frac{2}{\pi}\left[\frac{z^2}{2}+\frac{z^4}{1^2\cdot3^2\cdot4}+\frac{z^6}{1^2\cdot3^2\cdot5^2\cdot6}+\cdots\right]$$

12.2.8

$$\int_0^z[I_0(t)-\mathbf{L}_0(t)]\,dt-\frac{2}{\pi}[\ln(2z)+\gamma]$$

$$\sim-\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(2k)!\,(2k-1)!}{(k!)^2(2z)^{2k}}\quad(|\arg z|<\tfrac{1}{2}\pi)$$

12.2.9

$$\int_0^z\mathbf{L}_1(t)\,dt=\mathbf{L}_0(z)-\frac{2}{\pi}z$$

Relation to Modified Spherical Bessel Function

12.2.10

$$\mathbf{L}_{-(n+\frac{1}{2})}(z)=I_{(n+\frac{1}{2})}(z)\qquad(n\text{ an integer}\geq0)$$

12.3. Anger and Weber Functions

Anger's Function

12.3.1

$$\mathbf{J}_\nu(z)=\frac{1}{\pi}\int_0^{\pi}\cos(\nu\theta-z\sin\theta)\,d\theta$$

12.3.2

$$\mathbf{J}_n(z)=J_n(z)\qquad(n\text{ an integer})$$

Weber's Function

12.3.3

$$\mathbf{E}_\nu(z)=\frac{1}{\pi}\int_0^{\pi}\sin(\nu\theta-z\sin\theta)\,d\theta$$

Relations Between Anger's and Weber's Function

12.3.4

$$\sin(\nu\pi)\,\mathbf{J}_\nu(z)=\cos(\nu\pi)\,\mathbf{E}_\nu(z)-\mathbf{E}_{-\nu}(z)$$

12.3.5

$$\sin(\nu\pi)\,\mathbf{E}_\nu(z)=\mathbf{J}_{-\nu}(z)-\cos(\nu\pi)\,\mathbf{J}_\nu(z)$$

Relations Between Weber's Function and Struve's Function

If n is a positive integer or zero,

12.3.6

$$\mathbf{E}_n(z)=\frac{1}{\pi}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\frac{\Gamma(k+\frac{1}{2})(\frac{1}{2}z)^{n-2k-1}}{\Gamma(n+\frac{1}{2}-k)}-\mathbf{H}_n(z)$$ *

12.3.7

$$\mathbf{E}_{-n}(z)=\frac{(-1)^{n+1}}{\pi}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\frac{\Gamma(n-k-\frac{1}{2})(\frac{1}{2}z)^{-n+2k+1}}{\Gamma(k+\frac{3}{2})}-\mathbf{H}_{-n}(z)$$ *

*See page II.

12.3.8 $\mathbf{E}_0(z)=-\mathbf{H}_0(z)$

12.3.9 $\mathbf{E}_1(z)=\frac{2}{\pi}-\mathbf{H}_1(z)$

12.3.10 $\mathbf{E}_2(z)=\frac{2z}{3\pi}-\mathbf{H}_2(z)$

Numerical Methods

12.4. Use and Extension of the Tables

Example 1. Compute $\mathbf{L}_0(2)$ to 6D. From **Table 12.1** $I_0(2)-\mathbf{L}_0(2)=.342152$; from **Table 9.11** we have $I_0(2)=2.279585$ so that $\mathbf{L}_0(2)=1.937433$.

Example 2. Compute $\mathbf{H}_0(10)$ to 6D. From **Table 12.2** for $x^{-1}=.1$, $\mathbf{H}_0(10)-Y_0(10)=.063072$; from **Table 9.1** we have $Y_0(10)=.055671$. Thus, $\mathbf{H}_0(10)=.118743$.

Example 3. Compute $\int_0^x \mathbf{H}_0(t)dt$ for $x=6$ to 5D. Using **Tables 12.2, 11.1** and **4.2**, we have

$$\int_0^6 \mathbf{H}_0(t)dt=\int_0^6 Y_0(t)dt+\frac{2}{\pi}\ln 6+f_1(6)$$

$$=-.125951+(.636620)(1.791759)+.816764$$

$$=1.83148$$

Example 4. Compute $\mathbf{H}_n(x)$ for $x=4$, $-n=0(1)8$ to 6S. From **Table 12.1** we have $\mathbf{H}_0(4)=.1350146$, $\mathbf{H}_1(4)=1.0697267$. Using **12.1.9** we find

$\mathbf{H}_{-1}(4)=-.433107$	$\mathbf{H}_{-5}(4)=\ .689652$
$\mathbf{H}_{-2}(4)=\ .240694$	$\mathbf{H}_{-6}(4)=-1.21906$
$\mathbf{H}_{-3}(4)=\ .152624$	$\mathbf{H}_{-7}(4)=\ 2.82066$
$\mathbf{H}_{-4}(4)=-.439789$	$\mathbf{H}_{-8}(4)=-8.24933$

Example 5. Compute $\mathbf{H}_n(x)$ for $x=4$, $n=0(1)10$ to 7S. Starting with the values of $\mathbf{H}_0(4)$ and $\mathbf{H}_1(4)$ and using **12.1.9** with forward recurrence, we get

$\mathbf{H}_0(4)=\ .13501\ 46$	$\mathbf{H}_6(4)=.05433\ 54$
$\mathbf{H}_1(4)=1.06972\ 67$	$\mathbf{H}_7(4)=.01510\ 37$
$\mathbf{H}_2(4)=1.24867\ 51$	$\mathbf{H}_8(4)=.00367\ 33$
$\mathbf{H}_3(4)=\ .85800\ 95$	$\mathbf{H}_9(4)=.00080\ 02$
$\mathbf{H}_4(4)=\ .42637\ 41$	$\mathbf{H}_{10}(4)=.00018\ 25$
$\mathbf{H}_5(4)=\ .16719\ 87$	

We note that for $n>6$ there is a rapid loss of significant figures. On the other hand using **12.1.3** for $x=4$ we find $\mathbf{H}_9(4)=.0007935729$, $\mathbf{H}_{10}(4)=.00015447630$ and backward recurrence with **12.1.9** gives

$\mathbf{H}_8(4)=.00367\ 1495$	$\mathbf{H}_3(4)=\ .85800\ 94$
$\mathbf{H}_7(4)=.01510\ 315$	$\mathbf{H}_2(4)=1.24867\ 6$
$\mathbf{H}_6(4)=.05433\ 519$	$\mathbf{H}_1(4)=1.06972\ 7$
$\mathbf{H}_5(4)=.16719\ 87$	$\mathbf{H}_0(4)=\ .13501\ 4$
$\mathbf{H}_4(4)=.42637\ 43$	

Example 6. Compute $\mathbf{L}_n(.5)$ for $n=0(1)5$ to 8S. From **12.2.1** we find $\mathbf{L}_5(.5)=9.6307462\times10^{-7}$, $\mathbf{L}_4(.5)=2.1212342\times10^{-5}$. Then, with **12.2.4** we get

$\mathbf{L}_3(.5)=3.82465\ 03\times10^{-4}$	$\mathbf{L}_1(.5)=.05394\ 2181$
$\mathbf{L}_2(.5)=5.36867\ 34\times10^{-3}$	$\mathbf{L}_0(.5)=.32724\ 068$

Example 7. Compute $\mathbf{L}_n(.5)$ for $-n=0(1)5$ to 6S. From **Tables 12.1** and **9.8** we find $\mathbf{L}_0(.5)=.327240$, $\mathbf{L}_1(.5)=.053942$. Then employing **12.2.4** with backward recurrence we get

$\mathbf{L}_{-1}(.5)=\ .690562$	$\mathbf{L}_{-4}(.5)=-75.1418$
$\mathbf{L}_{-2}(.5)=-1.16177$	$\mathbf{L}_{-5}(.5)=1056.92$
$\mathbf{L}_{-3}(.5)=\ 7.43824$	

Example 8. Compute $\mathbf{L}_n(x)$ for $x=6$ and $-n=0(1)6$ to 8S. From **Tables 12.2** and **9.8** we find $\mathbf{L}_0(6)=67.124454$, $\mathbf{L}_1(6)=60.725011$. Using **12.2.4** we get

$\mathbf{L}_{-1}(6)=61.361631$	$\mathbf{L}_{-4}(6)=16.626028$
$\mathbf{L}_{-2}(6)=46.776680$	$\mathbf{L}_{-5}(6)=\ 7.984089$
$\mathbf{L}_{-3}(6)=30.159494$	$\mathbf{L}_{-6}(6)=\ 3.32780$

We note that there is no essential loss of accuracy until $n=-6$. However, if further values were necessary the recurrence procedure becomes unstable. To avoid the instability use the methods described in **Examples 5** and **6.**

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Texts

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Tables

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STRUVE FUNCTIONS

Table 12.1

x	$\mathbf{H}_0(x)$	$\mathbf{H}_1(x)$	$\int_0^x \mathbf{H}_0(t)dt$	$I_0(x)-\mathbf{L}_0(x)$	$I_1(x)-\mathbf{L}_1(x)$	$f_0(x)$	$\frac{2}{\pi}\int_x^\infty \frac{\mathbf{H}_0(t)}{t}dt$
0.0	0.00000 00	0.00000 00	0.000000	1.000000	0.000000	0.00000	1.000000
0.1	0.06359 13	0.00212 07	0.003181	0.938769	0.047939	0.09690	0.959487
0.2	0.12675 90	0.00846 57	0.012704	0.882134	0.091990	0.18791	0.919063
0.3	0.18908 29	0.01898 43	0.028505	0.829724	0.132480	0.27347	0.878819
0.4	0.25014 97	0.03359 25	0.050479	0.781198	0.169710	0.35398	0.838843
0.5	0.30955 59	0.05217 37	0.078480	0.736243	0.203952	0.42982	0.799223
0.6	0.36691 14	0.07457 97	0.112322	0.694573	0.235457	0.50134	0.760044
0.7	0.42184 24	0.10063 17	0.151781	0.655927	0.264454	0.56884	0.721389
0.8	0.47399 44	0.13012 25	0.196597	0.620063	0.291151	0.63262	0.683341
0.9	0.52303 50	0.16281 75	0.246476	0.586763	0.315740	0.69294	0.645976
1.0	0.56865 66	0.19845 73	0.301090	0.555823	0.338395	0.75005	0.609371
1.1	0.61057 87	0.23675 97	0.360084	0.527058	0.359276	0.80418	0.573596
1.2	0.64855 00	0.27742 18	0.423074	0.500300	0.378530	0.85553	0.538719
1.3	0.68235 03	0.32012 31	0.489655	0.475391	0.396290	0.90430	0.504803
1.4	0.71179 25	0.36452 80	0.559399	0.452188	0.412679	0.95066	0.471907
1.5	0.73672 35	0.41028 85	0.631863	0.430561	0.427810	0.99479	0.440086
1.6	0.75702 55	0.45704 72	0.706590	0.410388	0.441783	1.03682	0.409388
1.7	0.77261 68	0.50444 07	0.783111	0.391558	0.454694	1.07691	0.379857
1.8	0.78345 23	0.55210 21	0.860954	0.373970	0.466629	1.11518	0.351533
1.9	0.78952 36	0.59966 45	0.939643	0.357530	0.477666	1.15174	0.324450
2.0	0.79085 88	0.64676 37	1.018701	0.342152	0.487877	1.18672	0.298634
2.1	0.78752 22	0.69304 18	1.097659	0.327756	0.497329	1.22020	0.274109
2.2	0.77961 35	0.73814 96	1.176053	0.314270	0.506083	1.25230	0.250891
2.3	0.76726 65	0.78174 98	1.253434	0.301627	0.514194	1.28309	0.228992
2.4	0.75064 85	0.82351 98	1.329364	0.289765	0.521712	1.31265	0.208417
2.5	0.72995 77	0.86315 42	1.403427	0.278627	0.528685	1.34106	0.189168
2.6	0.70542 23	0.90036 74	1.475227	0.268162	0.535156	1.36840	0.171238
2.7	0.67729 77	0.93489 57	1.544392	0.258319	0.541164	1.39472	0.154618
2.8	0.64586 46	0.96649 98	1.610577	0.249056	0.546746	1.42008	0.139293
2.9	0.61142 64	0.99496 63	1.673465	0.240332	0.551933	1.44455	0.125242
3.0	0.57430 61	1.02010 96	1.732773	0.232107	0.556757	1.46816	0.112439
3.1	0.53484 44	1.04177 30	1.788248	0.224348	0.561246	1.49098	0.100857
3.2	0.49339 57	1.05983 03	1.839675	0.217022	0.565426	1.51305	0.090460
3.3	0.45032 57	1.07418 63	1.886873	0.210099	0.569319	1.53440	0.081212
3.4	0.40600 80	1.08477 74	1.929699	0.203553	0.572948	1.55508	0.073071
3.5	0.36082 08	1.09157 23	1.968046	0.197357	0.576333	1.57512	0.065992
3.6	0.31514 40	1.09457 16	2.001847	0.191488	0.579492	1.59456	0.059928
3.7	0.26935 59	1.09380 77	2.031071	0.185924	0.582442	1.61343	0.054829
3.8	0.22382 98	1.08934 44	2.055726	0.180646	0.585199	1.63176	0.050642
3.9	0.17893 12	1.08127 62	2.075858	0.175634	0.587776	1.64957	0.047311
4.0	0.13501 46	1.06972 67	2.091545	0.170872	0.590187	1.66689	0.044781
4.1	0.09242 08	1.05484 79	2.102905	0.166343	0.592445	1.68375	0.042994
4.2	0.05147 40	1.03681 86	2.110084	0.162032	0.594560	1.70017	0.041891
4.3	+0.01247 93	1.01584 22	2.113265	0.157926	0.596542	1.71616	0.041414
4.4	-0.02427 98	0.99214 51	2.112655	0.154012	0.598402	1.73176	0.041502
4.5	-0.05854 33	0.96597 44	2.108492	0.150279	0.600147	1.74697	0.042096
4.6	-0.09007 71	0.93759 56	2.101037	0.146714	0.601787	1.76182	0.043139
4.7	-0.11867 42	0.90729 01	2.090574	0.143309	0.603328	1.77632	0.044571
4.8	-0.14415 67	0.87535 28	2.077406	0.140053	0.604777	1.79049	0.046335
4.9	-0.16637 66	0.84208 90	2.061852	0.136938	0.606142	1.80434	0.048376
5.0	-0.18521 68	0.80781 19	2.044244	0.133955	0.607426	1.81788	0.050640
	$\left[\begin{matrix}(-4)6\\5\end{matrix}\right]$	$\left[\begin{matrix}(-4)5\\5\end{matrix}\right]$	$\left[\begin{matrix}(-4)8\\5\end{matrix}\right]$	$\left[\begin{matrix}(-4)6\\5\end{matrix}\right]$	$\left[\begin{matrix}(-4)5\\5\end{matrix}\right]$	$\left[\begin{matrix}(-4)7\\4\end{matrix}\right]$	$\left[\begin{matrix}(-4)2\\4\end{matrix}\right]$

$$\int_0^x [I_0(t)-\mathbf{L}_0(t)]dt = f_0(x)$$

$\mathbf{H}_0(x)$, $\mathbf{H}_1(x)$, $\mathbf{L}_0(x)$, $\mathbf{L}_1(x)$, compiled from Mathematical Tables Project, Table of the Struve functions $\mathbf{L}_\nu(x)$ and $\mathbf{H}_\nu(x)$, J. Math. Phys. **25**, 252–259, 1946 (with permission).

$\int_0^x \mathbf{H}_0(t)dt$, $\int_0^x [I_0(t)-\mathbf{L}_0(t)]dt$, $\frac{2}{\pi}\int_x^\infty \frac{\mathbf{H}_0(t)}{t}dt$, compiled from M. Abramowitz, Tables of integrals of Struve functions, J. Math. Phys. **29**, 49–51, 1950 (with permission).

Table 12.2 STRUVE FUNCTIONS FOR LARGE ARGUMENTS

x^{-1}	$\mathbf{H}_0(x)-Y_0(x)$	$\mathbf{H}_1(x)-Y_1(x)$	$f_1(x)$	$I_0(x)-\mathbf{L}_0(x)$	$I_1(x)-\mathbf{L}_1(x)$	$f_2(x)$	$f_3(x)$	$\langle x\rangle$
0.20	0.123301	0.659949	0.819924	0.133955	0.607426	0.793280	0.125868	5
0.19	0.117449	0.657819	0.818935	0.126683	0.610467	0.794902	0.119694	5
0.18	0.111556	0.655774	0.817981	0.119468	0.613348	0.796448	0.113505	6
0.17	0.105625	0.653818	0.817062	0.112319	0.616060	0.797910	0.107299	6
0.16	0.099655	0.651952	0.816182	0.105242	0.618598	0.799279	0.101079	6
0.15	0.093647	0.650180	0.815341	0.098241	0.620955	0.800551	0.094843	7
0.14	0.087602	0.648504	0.814541	0.091318	0.623129	0.801721	0.088593	7
0.13	0.081521	0.646927	0.813785	0.084474	0.625119	0.802787	0.082328	8
0.12	0.075404	0.645452	0.813074	0.077706	0.626927	0.803750	0.076051	8
0.11	0.069254	0.644081	0.812411	0.071010	0.628558	0.804611	0.069761	9
0.10	0.063072	0.642817	0.811796	0.064379	0.630018	0.805374	0.063460	10
0.09	0.056860	0.641663	0.811232	0.057805	0.631315	0.806047	0.057147	11
0.08	0.050620	0.640622	0.810722	0.051279	0.632457	0.806634	0.050824	13
0.07	0.044354	0.639696	0.810266	0.044793	0.633450	0.807140	0.044492	14
0.06	0.038064	0.638888	0.809866	0.038340	0.634302	0.807572	0.038152	17
0.05	0.031753	0.638200	0.809525	0.031912	0.635016	0.807933	0.031805	20
0.04	0.025425	0.637634	0.809244	0.025506	0.635596	0.808225	0.025451	25
0.03	0.019082	0.637191	0.809023	0.019116	0.636045	0.808450	0.019093	33
0.02	0.012727	0.636874	0.808865	0.012738	0.636365	0.808611	0.012731	50
0.01	0.006366	0.636683	0.808770	0.006367	0.636556	0.808706	0.006366	100
0.00	0.000000	0.636620	0.808738	0.000000	0.636620	0.808738	0.000000	∞
	$\left[\begin{smallmatrix}(-6)5\\3\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-5)2\\3\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-6)8\\3\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-5)1\\3\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-5)2\\3\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-5)1\\3\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-6)2\\3\end{smallmatrix}\right]$	

$$\int_0^x [\mathbf{H}_0(t)-Y_0(t)]dt=\frac{2}{\pi}\ln x+f_1(x)$$

$$\int_0^x [\mathbf{L}_0(t)-I_0(t)]dt=\frac{2}{\pi}\ln x+f_2(x)$$

$$\int_x^\infty \left[\frac{\mathbf{H}_0(t)-Y_0(t)}{t}\right]dt=f_3(x)$$

$\langle x\rangle$ = nearest integer to x.

Starting with $\mathbf{H}_0(x)$ and $\mathbf{H}_1(x)$, recurrence formula **12.1.9** may be used to generate $\mathbf{H}_n(x)$ for $n<0$. As long as $n<x/2$ (approx.), $\mathbf{H}_n(x)$ may be generated by forward recurrence. When $n>x/2$, forward recurrence is unstable. To avoid the instability, choose $n>>x$, compute $\mathbf{H}_k(x)$ and $\mathbf{H}_{k+1}(x)$ with **12.1.3**, and then use backward recurrence with **12.1.9**.

If $n>0$, $\mathbf{L}_n(x)$ must be generated by backward recurrence. If $n<0$, $\mathbf{L}_n(x)$ may be generated by backward recurrence as long as $\mathbf{L}_n(x)$ increases. If $n<0$ and $\mathbf{L}_n(x)$ is decreasing, forward recurrence should be used.

See Examples 4–8.